Grade 11 Unit 6 - Rotational Motion

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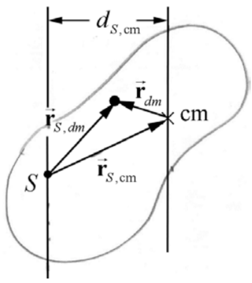
Parallel Axis Theorem and Rotational Dynamics

Usually, it is common to do the math while rotating bodies about an axis passing through their centers of masses. However, it might not always be the case. We have seen previously how to find the moment of inertia while being rotated through different axes, but now, we will see how we can easily determine the moment of inertia of a body through an axis parallel to the one passing through the center of mass.



$$\overrightarrow{\mathbf{r}}_{S,dm} = \overrightarrow{\mathbf{r}}_{S,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{dm}$$

$$\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},\perp,dm}\right| = r_{\perp,dm}$$

$$\left|\overrightarrow{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm}$$

$$\left|\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}}\right| = d_{S,\mathrm{cm}}$$

$$\overrightarrow{\mathbf{r}}_{S,\perp,dm} = \overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}$$
$$\overrightarrow{\mathbf{r}}_{S,\parallel,dm} = \overrightarrow{\mathbf{r}}_{S,\parallel,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\parallel,dm}$$

$$I_S = \int_{\mathrm{body}} dm\,(r_{S,\perp,dm})^2$$

$$\begin{aligned}(r_{S,\perp,dm})^2 &= \overrightarrow{\mathbf{r}}_{S,\perp,dm} \cdot \overrightarrow{\mathbf{r}}_{S,\perp,dm} \\ &= \left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}\right) \cdot \left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \overrightarrow{\mathbf{r}}_{\perp,dm}\right) \\ &= d_{S,\mathrm{cm}}^2 + (r_{\perp,dm})^2 + 2\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\end{aligned}$$

$$I_S = \int_{\mathrm{body}} dm d_{S,\mathrm{cm}}^2 + \int_{\mathrm{body}} dm\,(r_{\perp,dm})^2 + 2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right)$$

$$d_{S,\mathrm{cm}}^2 \int_{\mathrm{body}} dm = m d_{S,\mathrm{cm}}^2$$

$$I_{\mathrm{cm}} = \int_{\mathrm{body}} dm\,(r_{\perp,dm})^2$$

$$2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right) = \overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot 2\int_{\mathrm{body}} dm\overrightarrow{\mathbf{r}}_{\perp,dm}$$

$$2\int_{\mathrm{body}} dm\left(\overrightarrow{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \overrightarrow{\mathbf{r}}_{\perp,dm}\right) = 0$$

$$I_S = I_{\mathrm{cm}} + m d_{S,\mathrm{cm}}^2$$

This proof tells us that if we know the moment of inertia of a rigid body rotating through its center of mass, we can determine its moment of inertia through any axis parallel to the axis passing through the center of mass.

For example, let's see a rod rotating on one end instead of its center of mass. We have seen that for a rod rotating through its center of mass, its moment of inertia is given by:

$$I_{cm} = \frac{ML^2}{12}$$

If we choose its rotation axis to be one end of the rod, that makes the distance from the axis through the center of mass to the axis through one end:

$$d = \frac{L}{2}$$

Using parallel axis theorem, we can determine that the moment of inertia is:

$$I_s = I_{cm} + Md^2$$

$$I_s = \frac{ML^2}{12} + M(\frac{L}{2})^2$$

$$I_s = \frac{ML^2}{3}$$

Angular Momentum

The angular momentum of a rigid object is defined as the product of the moment of inertia and the angular velocity(or the cross product between **r** and linear momentum). It is analogous to linear momentum and is subject to the fundamental constraints of the conservation of angular momentum principle if there is no external torque on the object.

$$\mathbf{L} = \mathbf{r} \times \mathbf{B} \text{ or}$$

$$\mathrm{L} = \mathrm{I}\omega$$

We have seen earlier that for an object rotating with a net torque acting on it,

$$\tau_{net} = I\alpha$$

We also know that a net torque bi-implies angular acceleration, thus:

$$\tau_{net} = I(\frac{\omega_f - \omega_i}{\Delta t})$$

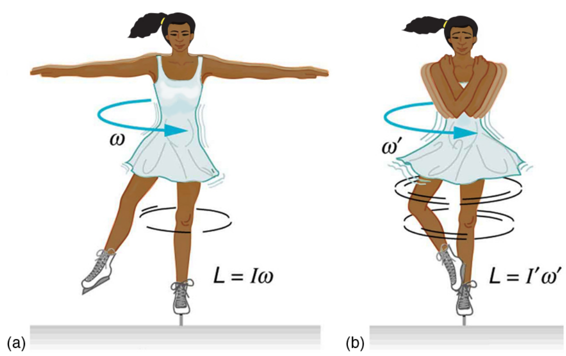
$$\tau_{net}\Delta t = I(\omega_f - \omega_i)$$

$$\tau_{net}\Delta t = \Delta I\omega$$

Thus,

$$\tau_{net}\Delta t = \Delta L$$

The quantity $\tau_{net}\Delta t$ is called the **angular impulse** of a body and is the rotational equivalent of impulse. We see that if the net torque on a rigid body is 0, then its change in angular momentum is 0 meaning angular momentum is conserved. Conservation of angular momentum is applicable in real life in multiple places. One case is with ballet-dancers where they can change how they are dancing and that, as a result, affecting how fast they are rotating.



An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is very small that it is negligible. In image (b), her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.